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QUADRATIC-SEPARABILITY OF BOOLEAN FUNCTIONS

BY

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled QUADRATIC-SEPARABILITY OF BOOLEAN FUNCTIONS, submitted by Fernando A. Amuchastegui, in partial fulfillment of the requirements for the degree of Master of Science.

ABSTRACT

Boolean functions which can be separated by a quadratic form are called quadratic boolean functions.

General properties of quadratic functions are investigated and a set of parameters which characterize a quadratic function are defined.

A set of theorems concerning the separability of unate functions are given and it is found that the class of quadratic functions is greater than the class of unate functions. For a given unate function a set of parameters is defined which enables us to find an ordering among the variables and will be used to find its realization.

Finally a method for realizing unate functions is given.

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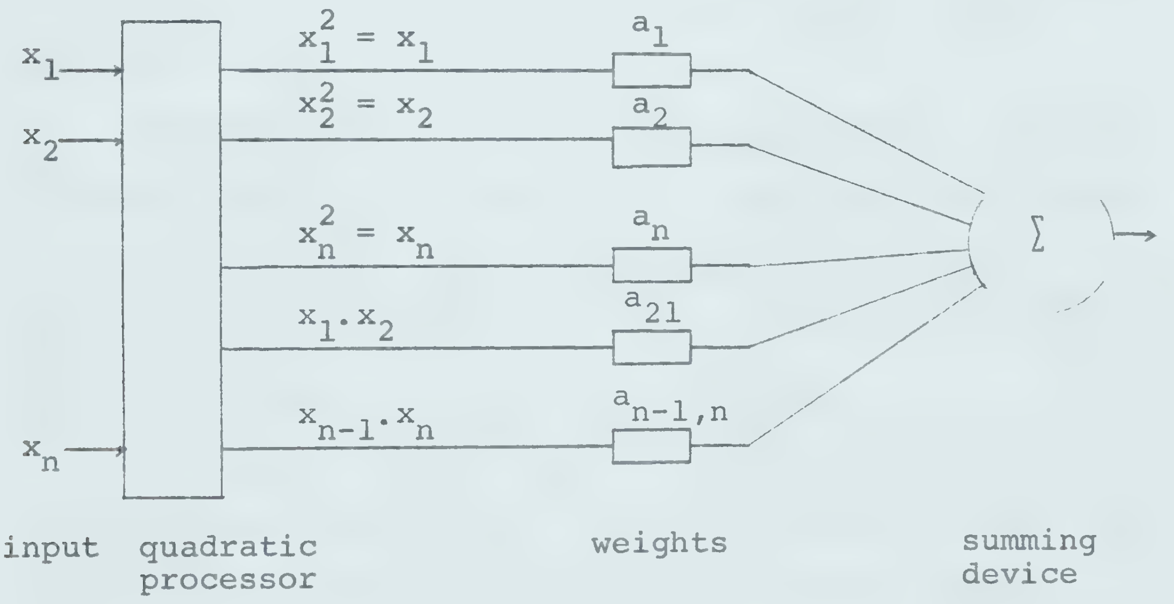
CHAPTER I : INTRODUCTION

In this thesis we are concerned with the realization of switching functions by means of what we call quadratic threshold elements (QTE). A quadratic threshold element has n external binary inputs $x_1, x_2, \ldots, x_n$, a single binary output, a single internal threshold T and $n(n+1)/2$ internal parameters or weights $a_1, a_2, \ldots, a_n, a_{21}, a_{31}, \ldots, a_{n,n-1}$, where a_i is related to the binary input x_i and a_{ij} is related to the binary inputs x_i and x_j.



If X_i is one of the 2^n different values of the ordered n-tuple $X = (x_1, x_2, \ldots, x_n)$ (Sheng [14]) and F is the switching function realized by the QTE, there exist a_{ij} such that

$$\text{(I)}\quad \begin{aligned} &\sum_{i=1}^{n} a_i x_i + \sum_{j=1}^{n-1} \sum_{i=j+1}^{n} a_{ij} x_i x_j \geq T \quad \text{iff} \quad F(X_i) = 1 \ , \\ &\sum_{i=1}^{n} a_i x_i + \sum_{j=1}^{m-1} \sum_{i=j+1}^{n} a_{ij} x_i x_j < T \quad \text{iff} \quad F(X_i) = 0 \ . \end{aligned}$$

QTE find applications in several fields. In learning machines, the quadratic discriminant functions are widely used. QTEs can also be thought as being a modified voting or majority logic, where some inputs can be related to each other, and then find application in digital data processing hardware (magnetic cores, resistor-transistor circuits) and in biological processes such as central nervous system of animals.

Although given a QTE we can always find a switching function F such that system (I) is satisfied, the converse is not true. What type of switching function can be realized by QTE and how to find the corresponding QT elements is the central problem in our work.

A problem closely related to quadratic separability is the linear separability of switching functions and has been subject of intensive research by, among others, Winder [15], Lewis and Coates [5], Sheng [14], Dertouzos [6] and Highleyman [9]. Linear separability turns out to be a special case of quadratic separability and solves many of the problems outlined before. However, the number of boolean functions that can be separated by using linear

threshold elements becomes progressively a smaller fraction of the total number of boolean functions as the number of binary inputs is increased. Using QTEs, the number of boolean functions that can be separated is substantially increased and, if the function is unate, we in fact show that it can always be realized with a QTE. In this thesis, we shall investigate the general properties of quadratic functions, then go on to the properties of unate functions and the general method of realization of unate functions. But before going any further, we need to define some basic definitions and concepts which will serve as the necessary background.

1.1 Boolean functions

Boolean algebra: A set B of elements $b_1, b_2, .., b_n$ is called a boolean algebra iff the following properties hold:

P1 : There are 2 binary operators (+), (.) which satisfy

$$b_i.b_i = b_i+b_i = b_i \qquad \text{(idempotent)}$$

$$b_i.b_j = b_j.b_i \;;\; b_i+b_i = b_j+b_i \qquad \text{(commutative)}$$

$$\left.\begin{array}{l} b_i.(b_j.b_k) = (b_i.b_j).b_k \\ b_i+(b_j+b_k) = (b_i.b_j)+b_k \end{array}\right. \qquad \text{(associative law)}$$

P2 : $$b_i.(b_i+b_j) = b_i+(b_i.b_j) = b_i \qquad \text{(absorption law)}$$

P3 : $b_i \cdot (b_j+b_k) = (b_i \cdot b_j)+(b_i \cdot b_k)$
$b_i+(b_j \cdot b_k) = (b_i+b_j) \cdot (b_i+b_k)$ (mutually distributive)

P4 : There exist universal bounds, 0 and I which satisfy

$0 \cdot b_i = 0,\ 0+b_i = b_i \ ;\ 1 \cdot b_i = b_i,\ 1+b_i = I$

P5 : There exists an element $\bar{b}_i$ for every b_i such that

$b_i+\bar{b}_i = I \ ;\ b_i \cdot \bar{b}_i = 0$ (complementation)

These properties hold for all $b_i \varepsilon B$.

<u>Boolean functions</u>: A function $F(x_1, x_2,\ldots,x_n)$ is called a boolean function of n variables (or more precisely, a n-ary truth function) when each of its variables x_i (i=1,n) can take only two values (0 or 1, True or False, a and $\bar{a}$ in general), and also the value of the function can be either 0 or 1, True or False, a or $\bar{a}$. The definition domain consists of all possible sequences X_i = $(x_1, x_2,\ldots, x_n)$ of length n each of whose element x_i (i=1,n) is either 0 or 1, True or False, a or $\bar{a}$. The function $F(x_1, x_2,\ldots, x_n)$ assigns a truth value (0 or 1, T or F, a or $\bar{a}$) to every element of the definition domain.

<u>Example 1.1</u>

Let F be $F = x_1+x_2$. The definition domains are the $2^n = 4$ points X = (0 0), (0 1), (1 0), (1 1) being the first one assigned the value of 0, and the rest of value 1.

The Space B^n

We will now give a different definition of boolean functions. Let us consider n binary variables where each variable can be either 0 or 1. The set of all possible combinations of values for these n variables can be obtained by forming the cartesian product of n identical sets $B = \{0,1\}$. This set of n-tuples is

$$B^n = B.B. \ldots\ldots B$$

The set B^n is then the set of all n-tuples of 0 and 1, and boolean functions can be defined as the mapping of the set B^n into the set B

$$B^n \rightarrow B$$

B^n is also called the n-cube and is a subset of the Euclidean n-space E^n.

From now on, each n-tuple will be designated by X_i $(i=1,2^n)$ and will be called a vertex of the n-cube. The set of all vertices will be designated by X.

1.2 Geometrical interpretation of boolean functions

Let us from now on work with the assumption that the variables x_i $(i=1,n)$ and the function $F(x_1,x_2,\ldots,x_n)$ are to take only the values 0 or 1.

For example, if we are working with two variables, then

x_1 can be 0 or 1

x_2 can be 0 or 1

We can represent this situation graphically, as shown in Fig 1.1

Fig 1.1 Graphical representation of B^2

We see that the values of any boolean function of two variables will be a mapping of the vertices of the square into 0 or 1. In three dimension (that is, with three variables) we have to refer to a cube in E^3.

<u>Example 1.2</u> $F_2 = x_1+x_2x_3$

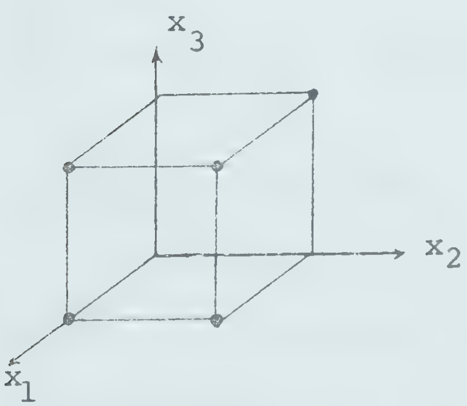


Fig 1.2 Vertices of F_2

<u>Table I</u>

Truth table of F_2

x_1	x_2	x_3	F(X)
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

In general, for a boolean function of n variables, we have to resort to an hypercube in E^n, or n-cube.

As we see from Fig 1.2 and Table I, we can make a one to one correspondence between the vertices in the n-cube and the minterms of n variables. A minterm or fundamental product is a product with all n variables included. For example, in $F_2 = x_1 + x_2x_3$ the minterms are $\bar{x}_1x_2x_3$, $x_1\bar{x}_2\bar{x}_3$, $x_1\bar{x}_2x_3$, $x_1x_2\bar{x}_3$ and $x_1x_2x_3$.

Definitions:

A simple product is a product of k variables ($1 \leq k \leq n$) where some variables can be complemented. A simple product R is called a prime implicant (PI) of $F(x_1,x_2,..,x_n)$ if $R \rightarrow F$ and there exists no other simple product R' such that $R \rightarrow R'$.

A function is said to be in irredundant normal form when it is represented as a sum of a subset of its PIs such that no PI is irredundant.

1.3 Reduced functions

Let $F(x_1,x_2,..,x_n)$ be an arbitrary boolean function in B^n. We define the reduced function of F with respect to x_i, and denote it by F_{x_i}, as being the function resulting from assigning to x_i the value of 1 (and to $\bar{x}_i$ the value of 0). That is

$$F_{x_i} = F(x_1,x_2,...,(x_i=1,\bar{x}_i=0),...,x_n)$$

In general, let S be a simple product. The reduced function of F with respect to S is gotten by assigning to S the value of 1.

Example 1.3

$$F = x_1x_3+\bar{x}_2x_4x_5+x_2x_5$$

$$S = x_1\bar{x}_2x_4$$

$$F_S = F(x_1=1,(x_2=0,\bar{x}_2=1),x_3,x_4=1,x_5)$$

$$F_S = x_3+x_5$$

In particular, we have the very useful expression

$$F = x_iF_{x_i}+\bar{x}_iF_{\bar{x}_i}$$

Let F_1 and F_2 be two boolean functions. We say that F_1 contains F_2, or

$$F_1 \supseteq F_2 \text{ , iff for any } X_i \quad F_2(X_i) = 1 \rightarrow F_1(X_i) = 1$$

If there exists some X_i such that $F_2(X_i) = 0$, $F_1(X_i) = 1$, then we say $F_1 \supset F_2$. We say that two boolean functions F_1 and F_2 are comparable, if $F_1 \supseteq F_2$ or $F_2 \supseteq F_1$.

Monotonicity

A function $F(x_1,x_2,..,x_n)$ is said to be 1-monotonic iff the reduced functions F_{x_i}, $F_{\bar{x}_i}$ (i=1,n) are comparable. A function $F(x_1,x_2,..,x_n)$ is called k-monotonic iff the

reduced functions along any i varables $0 \leq i \leq k$ are comparable.

A boolean function $F(x_1,x_2,..,x_n)$ is called unate iff it can be represented in irredundant normal form in which no variable appears both complemented and uncomplemented. It can be shown easily that any unate function is 1-monotonic (McNaughton, [12]).

1.4 Linear separability

Let $F(x_1,x_2,...,x_n)$ be a boolean function. We say that F is linearly separable iff there exist [A;T] such that:

$$A^T X_i \geq T \quad \text{iff} \quad F(X_i) = 1$$

and

$$A^T X_i < T \quad \text{iff} \quad F(X_i) = 0$$

where $A = (a_1,a_2,...,a_n)$ is a vector of n real parameters and T is a real number.

Example 1.4

$F_3 = x_1+x_2$. If we choose [A;T] to be A = (1,1), T = 1, then

x_1	x_2	F(X)	A^TX		
0	0	0	0	<	T
0	1	1	1	≥	T
1	0	1	1	≥	T
1	1	1	2	≥	T

We then say that the function $F_3 = x_1+x_2$ is linearly separable or threshold, and [A;T] is its (linear) realization. The equation $A^T X-T = x_1+x_2-1 = 0$ is the equation of a straight line in E^2 (see Fig 1.3) which divides the E^2 space into two regions: one containing the true vertices (F = 1) and the other the false vertices (F = 0).

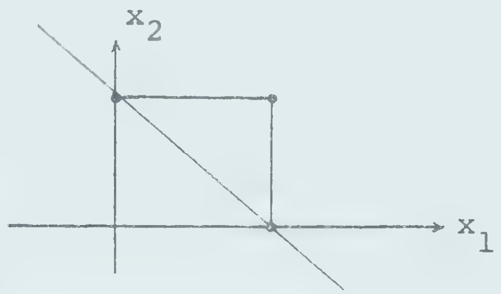


Fig 1.3 Geometrical interpretation of the threshold function F_3

In general, if $F(x_1,x_2,...,x_n)$ is linearly separable, then there exists an hyperplane $A^T X-T = 0$ in E^n which divides the B^n space in two regions; one containing the true vertices and the other the false vertices.

<u>Theorems relating to linear threshold functions</u>

There are several theorems relating to linear threshold functions and we will mention only one of them. Its proof can be found in Sheng [14], Lewis and Coates [5], McNaughton [12].

<u>Theorem 1</u>. Let $F = F(x_1,x_2,..,x_n)$ be a boolean function of n variables. If F is linearly separable

then it is n-monotonic.

<u>Corollary 1.1</u>. If F is a threshold function of n variables then F is unate.

CHAPTER II : QUADRATIC FUNCTIONS

2.1 Quadratic separability

Let $F = F(x_1 x_2 \ldots x_n)$ be a boolean function of n variables. We said that F is quadratically separable, iff there exist [A;T] such that

$$X^T A X \geq T \quad \text{iff} \quad F(X) = 1$$

and

$$X^T A X < T \quad \text{iff} \quad F(X) = 0$$

where

$$A = \left[\begin{array}{ccc|cc} a_{11} & 0 & & 0 & \\ a_{21} & a_{22} & & 0 & \\ a_{31} & & & 0 & \\ \hline a_{n1} & & & a_{nn} & \end{array}\right]$$

a_{ij} (j=1,2,..,n; i=j,j+1,..,n) and T are real numbers. The geometrical interpretation of quadratic functions (its name deriving from quadratic form) is that the points in the n-cube corresponding to F = 1 are separated from those corresponding to F = 0 by a quadratic surface in E^n.

If a boolean function is quadratically separable, then there exist $P(X) = X^T A X - T$ such that $P(X) \geq 0$ iff $F(X) = 1$ and $P(X) < 0$ iff $F(X) = 0$. Then $P(X) = 0$ is the separating surface. The quadratic form gives rise to

different figures, depending on the properties of the matrix A.

Case a:

The matrix A is diagonal. Then

$$P(X) = X^T A\ X - T = a_{11}x_1^2+a_{22}x_2^2+\ldots+a_{nn}x_n^2-T = 0$$

The separating surface is an ellipsoid whose semiaxes are coincident with the x_i's. Also, since $x_i^2 = x_i$ for $x_i = 0,1$ we see that

$$P_1(X) = a_{11}x_1+a_{22}x_2+\ldots+a_{nn}x_n-T = 0$$

is a separating hyperplane which also solves the problem. It is therefore clear that the threshold functions are a subclass of quadratic functions.

Example 2.1

$$F_1 = x_1x_2+x_1x_3+x_2x_3$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad ; \quad T = 2$$

Separating ellipsoid $x_1^2+x_2^2+x_3^2-2 = 0$

Separating hyperplane $x_1+x_2+x_3-2 = 0$

Fig 2.1 shows the graphical interpretation.

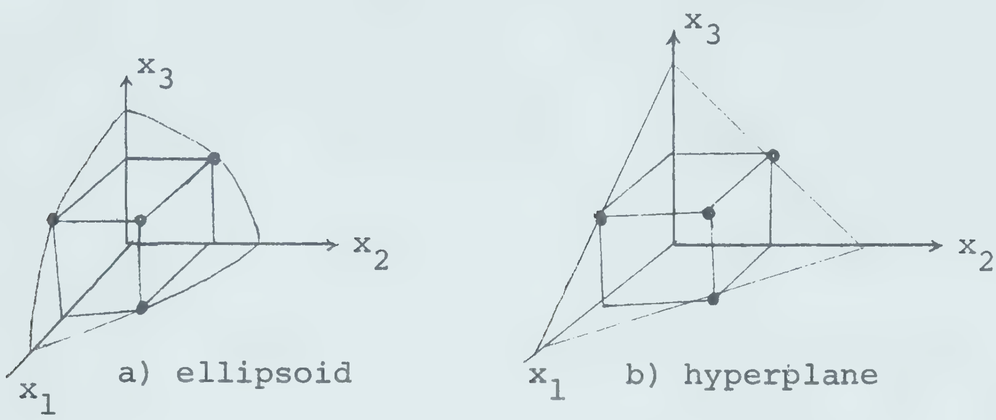


Fig 2.1 Separating surfaces for F_1

Case b:

The matrix A is positive definite, but not diagonal. Then $X^TAX-T = 0$ is the equation of an ellipsoid whose semiaxes are not coincident with the x_i's.

Example 2.2

$F_1 = x_1x_2+x_3x_4$ admits a realization [A;T], where

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad ; \qquad T = 3$$

and $P(X) = x_1^2+x_2^2+x_3^2+x_4^2+x_1x_2+x_3x_4-3 = 0$ is the separating ellipsoid.

The true vertices $(F=1)$ are outside $P(X) \geq 0$

The false vertices $(F=0)$ are inside $P(X) < 0$.

Case c:

The matrix is neither positive nor negative definite. Then $P(X) = X^T AX-T = 0$ is the equation of an ellipsoid, parabola, hyperbola or degen erate conic.

Example 2.3

$F_2 = x_1\bar{x}_2+\bar{x}_1x_2$. This function is not unate and hence cannot be separated by a linear function. We will solve the problem using three different matrices and showing the different separating surfaces.

Solution 1:

$$A = \begin{pmatrix} 2 & 0 \\ -3 & 2 \end{pmatrix} \quad ; \quad T = 2$$

Truth Table for F_2

x_1	x_2	F(X)	P(X)
0	0	0	$-2 < 0$
0	1	1	$0 \geq 0$
1	0	1	$0 \geq 0$
1	1	0	$-1 < 0$

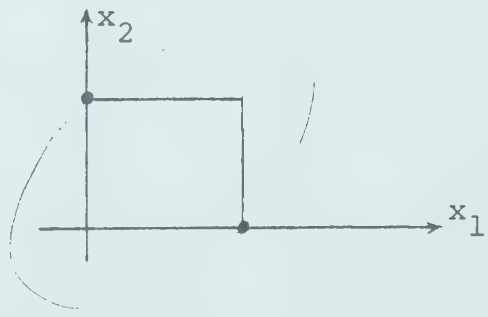


Fig 2.2 Separating ellipsoid

Solution 2:

$$A = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \quad ; \quad T = 1$$

$P(X) = x_1^2+x_2^2-2x_1x_2-1 = (x_1-x_2)^2-1 = 0$ and this can be factorized as $x_1 = x_2+1$; $x_1 = x_2-1$ (straight lines)

x_1 x_2	F(X)	P(X)
0 0	0	$-1 < 0$
0 1	1	$0 \geq 0$
1 0	1	$0 \geq 0$
1 1	0	$-1 < 0$

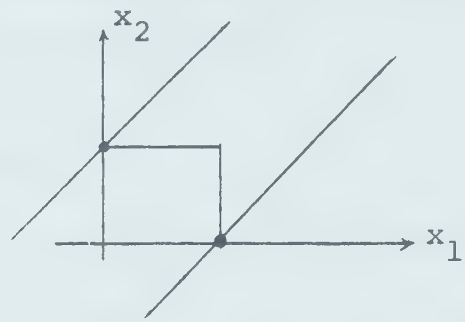


Fig 2.3 F_2 is separated by straight lines.

Solution 3:

$$A = \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix} \quad ; \quad T = 1$$

$P(X) = x_1^2+x_2^2-4x_1x_2-1 = 0$

x_1 x_2	F(X)	P(X)
0 0	0	$-1 < 0$
0 1	1	$0 \geq 0$
1 0	1	$0 \geq 0$
1 1	0	$-3 < 0$

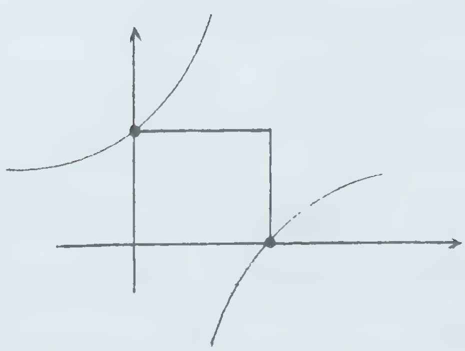


Fig 2.4 F_2 is separated by an hyperbola.

As we have seen, even when the function is not unate we may still be able to separate it.

A general method to realize non-unate functions has yet to be developed, however, for functions with not too many variables, a method based on trial-and-error may work even though it is not usually known in advance whether a given function is quadratic or not.

2.2 Generalized Chow parameters

Let $F(x_1x_2\ldots x_n)$ be a boolean function of n variables and let $M = m(F)$ $(0 \leq M \leq 2^n)$ be the number of vertices for which the function F is equal to 1, i.e. the number of true vertices $(0 \leq M \leq 2^n)$. Also let

$$c_i = m(F_{x_i}) \qquad (i=1,2,\ldots,n)$$

$$c_{ij} = m(F_{x_ix_j}) \qquad (j=1,2,\ldots,n-1;\ i=j+1,\ldots,n)$$

so that c_i is the number of true vertices with $x_i = 1$ and c_{ij} is the number of true vertices with $x_i=x_j=1$. Now, M, the number of true vertices, and c_i $(i=1,2,..,n)$ form a set of n+1 parameters which characterize (Sheng, p.20-21) the realization of a (linear) threshold function, and are called the Chow parameters. By extension, M, c_i and c_{ij} form a set of $1+n(n+1)/2$ parameters which, as will be shown later, completely characterize the realization of a

quadratic function, and are called in this thesis the generalized Chow parameters. We now set $c_{ii} = c_i$ $(i=1,n)$ and define the matrix C as follows:

$$C = \left(\begin{array}{cc|c} c_{11} & 0 & 0 \\ c_{21} & c_{22} & 0 \\ & & \\ \hline c_{n1} & & c_{nn} \end{array}\right)$$

and the generalized Chow parameters are then represented in compact form by M and C.

<u>Example 2.4</u>

$$F = x_1 + x_2 x_3$$

x_1	x_2	x_3	F
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

$M = m(F) = 5$

$c_1 = 4$; $c_2 = 3$; $c_3 = 3$;

$c_{21} = 2$; $c_{31} = 2$; $c_{32} = 2$.

$$C = \begin{pmatrix} 4 & 0 & 0 \\ 2 & 3 & 0 \\ 2 & 2 & 3 \end{pmatrix}$$

2.3 Properties of quadratic functions

Lemma 1:

If $F_1(X) = F_1(x_1,x_2,..,x_n)$ is a quadratic function with realization $[A';T']$, then $F_2(X') = F_1(X)+x_{n+1}$ $(X=(x_1,x_2,..,x_n),\ X'=(x_1,x_2,..,x_n,x_{n+1}))$ is also a quadratic function with realization $[A";T"]$ where $a''_{ij} = a'_{ij}$ $(i,j=1,n)$, $a''_{n+1,i}=0$ $(i=1,n)$, $a''_{n+1,n+1}=T_m-T_n$, $T"= T'$ and where $T_m= \max(X_i^TA'X_i)$ such that $F_1(X_i) = 1$ and $T_n = \min(X_i^TA'X_i)$ such that $F_1(X_i) = 0$.

Proof:

For $x_{n+1} = 0$

$$X_i'^TA"X_i' = X_i^TA'X_i + 0 \geq T' = T" \quad \text{iff} \quad F_1(X_i)=1 \rightarrow F_2(X_i')=1$$

$$X_i'^TA"X_i' = X_i^TA'X_i + 0 < T' = T" \quad \text{iff} \quad F_1(X_i)=0 \rightarrow F_2(X_i')=0$$

For $x_{n+1} = 1$

$$X_i'^TA"X_A' = X_i^TA'X_i+a_{n+1,n+1} = X_i^TA'X_i+(T_m-T_n) \geq T_m \geq T' = T"$$

and $F_2(X_i') = 1$ since $x_{n+1} = 1$

Lemma 2:

If $F_1(X) = F_1(x_1,x_2,..,x_n)$ is a quadratic function with realization $[A';T']$, then $F_2(X') = F_1(X).x_{n+1}$ $(X=(x_1,x_2,..,x_n),\ X'=(x_1,x_2,..,x_n,x_{n+1}))$ is also a quadratic function with realization $[A";T"]$ where

$a''_{ij} = a'_{ij}$ $(i,j=1,n)$, $a''_{n+1,i} = 0$ $(i=1,n)$, $a_{n+1,n+1} > (T_m - T_n)$, $T'' = T_n + a_{n+1,n+1}$ and where $T_m = \max(X_i^T A' X_i)$ such that $F_1(X_i) = 1$ and $T_n = \min(X_i^T A' X_i)$ such that $F_1(X_i) = 1$.

Proof:

For $x_{n+1} = 1$

$$X_i'^T A'' X_i' = X_i^T A' X_i + a_{n+1,n+1} \geq T_n + a_{n+1,n+1} = T'' \quad \text{iff}$$

$$F_1(X_i) = 1 \rightarrow F_2(X_i') = 1$$

$$X_i'^T A'' X_i' = X_i^T A' X_i + a_{n+1,n+1} < T_n + a_{n+1,n+1} = T'' \quad \text{iff}$$

$$F_1(X_i) = 0 \rightarrow F_2(X_i') = 0$$

For $x_{n+1} = 0$

$$X_i'^T A'' X_i' = X_i^T A' X_i + 0 \leq T_m < T_n + a_{n+1,n+1} = T'' \quad \text{and} \quad F_2(X_i') = 0 .$$

Lemma 3:

If F_1, F_2 are two quadratic functions, then $F_1 + F_2$ is not necessarily a quadratic function.

Proof:

$$F_1 = x_1 x_2 x_3 + x_1 \bar{x}_2 \bar{x}_3 \quad , \quad F_2 = \bar{x}_1 x_2 \bar{x}_3 + \bar{x}_1 \bar{x}_2 x_3$$

are quadratic functions with realizations

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 2 & -1 \end{pmatrix} \; ; \quad T_1 = 3 \; ; \quad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -2 & 1 \end{pmatrix} \; ; \quad T_2 = 0$$

but the function $F = F_1+F_2$ (see Fig 2.5) is not quadratic, because that would imply the existence of A,T such that

$$X^T A\ X - T \geq 0 \qquad \text{iff} \qquad F(X) = 1$$

$$X^T A\ X - T < 0 \qquad \text{iff} \qquad F(X) = 0$$

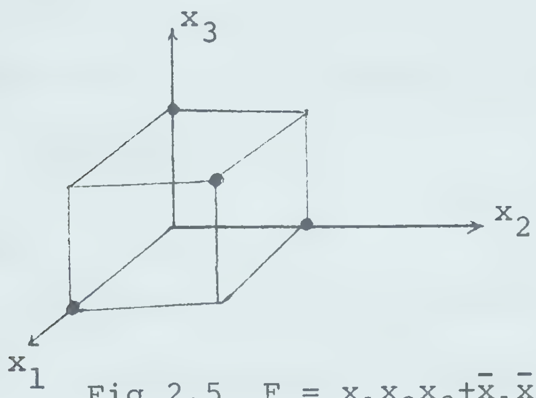


Fig 2.5 $F = x_1x_2x_3+\bar{x}_1\bar{x}_2x_3+\bar{x}_1x_2\bar{x}_3+x_1\bar{x}_2\bar{x}_3$

In our case $F = F_1+F_2 = x_1x_2x_3+\bar{x}_1\bar{x}_2x_3+\bar{x}_1x_2\bar{x}_3+x_1\bar{x}_2\bar{x}_3$

$x_1x_2x_3$	F(X)	$X^TA\ X$	
0 0 0	0	$0 < T$	I
0 1 1	0	$a_2+a_3+a_{32} < T$	II
1 0 1	0	$a_1+a_3+a_{31} < T$	III
1 1 0	0	$a_1+a_2+a_{21} < T$	IV
0 0 1	1	$a_3 \geq T$	V
0 1 0	1	$a_2 \geq T$	VI
1 0 0	1	$a_1 \geq T$	VII
1 1 1	1	$a_1+a_2+a_3+a_{21}+a_{31}+a_{32} \geq T$	VIII

Adding II, III and IV we obtain

$$2(a_1+a_2+a_3)+a_{21}+a_{32}+a_{32} < 3T \quad ,$$

whereas the addition of V, VI, VII and VIII results in

$$2(a_1+a_2+a_3)+a_{21}+a_{32}+a_{32} \geq 3T \quad .$$

The equations are contradictory, therefore the function is not quadratic.

Lemma 4:

If F_1 and F_2 are two quadratic functions, then F_1 $F_1 \cdot F_2$ is not necessarily a quadratic function.

Proof:

$$F_1 = x_3+x_1\bar{x}_2+\bar{x}_1x_2$$

and $F_2 = \bar{x}_3+\bar{x}_1\bar{x}_2+x_1x_2$ are quadratic functions with realizations $[A_1;T_1]$, $[A_2;T_2]$, where

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad T_1 = 1 \quad ; \quad A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ -1 & -1 & 0 \end{pmatrix} \quad T_2 = 0$$

and $F = F_1 \cdot F_2 = x_1x_2x_3+\bar{x}_1\bar{x}_2x_3+\bar{x}_1x_2\bar{x}_3+x_1\bar{x}_2\bar{x}_3$ is the same function F as in Example 2.5, and hence not quadratic.

2.4 Admissible transformations

Given the n-cube B^n, the number of different boolean functions that can be formed is 2^{2^n}, which, for large

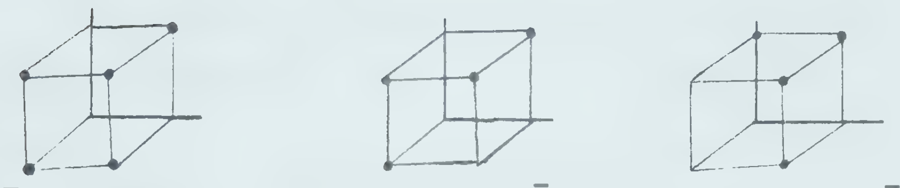
values of n, becomes very large. However, let us analyse the functions for the following example:

<u>Example 2.5</u>:

$$F_1 = x_1+x_2x_3$$

$$F_2 = x_3+x_1\bar{x}_2$$

$$F_3 = x_1x_2+\bar{x}_1x_3$$



$F_1=x_1+x_2x_3$ $\qquad$ $F_2=x_3+x_1\bar{x}_2$ $\qquad$ $F_3=x_1x_2+\bar{x}_1x_3$

Fig 2.6

We see from Fig 2.6 that F_1 and F_2 have the same "shape" and both differ from F_3 in that aspect. That is, the geometrical configuration of F_1 is identical to that of F_2.

We have seen in Chapter I that if a function is threshold, then there exists an hyperplane which divides the n-cube into two regions, one containing the true vertices and the other the false vertices. Similarly, if a function is quadratic then there exists a quadratic form, or conic, in E^n which divides the n-cube in regions (2 or 3), one (s) containing the true vertices and the other (s) the false vertices.

In both cases, the problem of linear or quadratical separability can be thought as being a geometrical one, and we are therefore interested in dividing the 2^{2^n} boolean functions into disjoint classes, each class consisting of different boolean functions but with the same geometrical configuration. These classes are called equivalence classes.

Definition:

We define the equivalence relation between two functions as follows:

$f \approx g$ (f equivalent to g)

if there exists a function $Q(X)$ such that $f(X) = f(Q(X))$ where Q consists only of the permutation and complementation of variables.

Example 2.6:

Given F_1 and F_2 from Example 2.5

$F_1 = x_1+x_2x_3$

$F_2 = x_3+x_1\bar{x}_2$

and with Q such that $x_1\ x_2$, $\bar{x}_2\ x_3$, $x_3\ x_1$ we have

$F_2(Q(X)) = x_1+x_2x_3 = F_1$.

Theorems relating to admissible transformations

Theorem 2.1:

If $F_1 = F_1(x_1,x_2,..,x_i,..,x_n)$ is a quadratic function with realization [A';T'], then $F_2 = F_1(x_1,x_2,..,\bar{x}_i,..,x_n)$ is also a quadratic function with realization [A";T"] where $T" = T'-a_i'$ and A" is such that $a_i" = -a_i'$; $a_k" = a_k'+a_{ik}'$, $a_{ik}" = -a_{ik}'$ $(k=1,2,..,n;k\neq i)$.

Proof:

If [A';T'] is the realization of $F_1(x_1,x_2,..,x_i,..,x_n)$ then for the true vertices $X_i^T A'X_i \geq T'$. Expanding $X_i^T A'X_i$:

$$(*)\quad X_i^T A'X_i = SUM + a_i'x_i + \sum_{k=1}^{i-1} a_{ik}'x_ix_k + \sum_{k=i+1}^{n} a_{ki}'x_ix_k \geq T'$$

where SUM is the part of $X_i^T A'X_i$ which does not depend on x_i. For F_2 we have that the relationship (*) has to hold when we change x_i to $(1-x_i)$, that is, we must show

$$(**)\quad SUM + a_i'(1-x_i) + \sum_{k=1}^{i-1} a_{ik}'(1-x_i)x_k + \sum_{k=i+1}^{n} a_{ki}'(1-x_i)x_k \geq T'$$

But (**) may be written as

$$SUM + (-a_i')x_i + \sum_{k=1}^{i-1} a_{ik}'x_k + \sum_{k=1}^{i-1} (-a_{ik}')x_ix_k$$

$$+ \sum_{k=i+1}^{m} (-a_{ki}')x_ix_k \geq T'-a_i'$$

and looking to the coefficients of x_i, x_k, $x_i x_k$ and to the second member, we get

$$T'' = T' - a_i'$$

$$\left.\begin{array}{l} a_{ki}'' = -a_{ki}' \\ \\ a_k'' = a_k' + a_{ki}' \end{array}\right\} \qquad k = i+1,n$$

$$\left.\begin{array}{l} a_{ij}'' = -a_{ij} \\ \\ a_j'' = a_j' + a_{ij}' \end{array}\right\} \qquad j = 1,i-1$$

$$a_i'' = -a_i$$

<u>Example 2.7</u>:

$F = x_1x_2+x_3x_4$ admits a realization of

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad ; \quad T = 3$$

The realization for $F_1 = x_1\bar{x}_2+x_3x_4$ will be then

$T' = T - a_2 = 3-1 = 2$;

$a_1' = a_1 + a_{21} = 1+1 = 2$;

$a_2' = -a_2 = -1$;

$a_{21}' = -a_{21} = -1$;

and the others will not be affected.

$x_1x_2x_3x_4$	$F_1(X)$	$X^TA\ X$
0 0 0 0	0	0
0 0 0 1	0	1
0 0 1 0	0	1
0 0 1 1	1	3
0 0 0 0	0	-1
0 1 0 1	0	0
0 1 1 0	0	0
0 1 1 1	1	2
1 0 0 0	1	2
1 0 0 1	1	3
1 0 1 0	1	3
1 0 1 1	1	5
1 1 0 0	0	0
1 1 0 1	0	1
1 1 1 0	0	1
1 1 1 1	1	3

$$A' = \begin{pmatrix} 2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$T' = 2$

The above formula can easily be extended to k variables ($0 \le k \le n$).

<u>Theorem 2.2</u>:

If $F(x_1,x_2,..,x_n)$ is a quadratic function, then $\bar{F}$ is also a quadratic function.

<u>Proof</u>:

Since F is a quadratic function there exist T_μ, T_ℓ and A such that $T_\mu > T_\ell$ and

$$X_i^T A\ X_i > T_\mu \qquad \text{iff} \qquad F(X_i) = 1$$

$$X_i^T A\ X_i < T_\ell \qquad \text{iff} \qquad F(X_i) = 0$$

and that can be written as

$$X_i^T(-A)\ X_i < (-T_\mu) \ \text{iff} \qquad F(X_i) = 1$$

$$X_i^T(-A)\ X_i > (-T_\ell) \ \text{iff} \qquad F(X_i) = 0$$

If we now put $G = \bar{F}$; $-T_\mu = T_k$; $-T_\ell = T_m$; $-A = A'$, we have

$$X_i^T A' X_i > T_m \qquad \text{iff} \qquad G(X_i) = 1$$

$$X_i^T A' X_i \le T_k \qquad \text{iff} \qquad G(X_i) = 0$$

that is, $\bar{F} = G$ is also a quadratic function.

<u>Theorem 2.3</u>:

If $F = F(x_1,x_2,..,x_i,..,x_j,..,x_n)$ is a quadratic function with realization [A;T], then $F_1 = F(x_1,x_2,..,x_j,..,x_i.,x_n)$ is also a quadratic function with realization $[A_1;T_1]$,

where $T_1 = T$ and A_1 such that $a'_{ik} = a_{jk}$, $a'_{jk} = a_{ik}$ $(i,j=1,2,..,n)$. The proof is obvious and is omitted here.

Theorems 2.1 and 2.3 prove that the existence of a quadratic realization for a boolean function is not affected by permutation and/or complementation of its variables, as was expressed at the beginning of 2.4.

CHAPTER III : UNATE QUADRATIC FUNCTIONS

Definitions

As mentioned in Chapter I, a unate function is a boolean function that can be represented by an irredundant normal form in which no variable appears both complemented and uncomplemented. A function is positive in an argument x_i iff it can be represented by an irredundant normal form in which x_i appears uncomplemented. More generally, a function is said to be positive iff it is positive in all of its arguments. A function can be expressed as

$$F = x_i F_{x_i} + \bar{x}_i F_{\bar{x}_i} \quad .$$

If the function is positive in x_i, then (Sheng [14], Lewis and Coates [5]) $F_{x_i} \Rightarrow F_{\bar{x}_i}$ and $F = x_i F_{x_i} + F_{\bar{x}_i}$.

Geometrical interpretation of unate functions

A unate boolean function of n variables is represented in the usual manner, as being the mapping of the vertices of the n-cube into 0 or 1 value. At this point, it is useful to think of a partial ordering among the 2^n vertices of the n-cube.

Let

$$X_1 = x_1^1 \, x_2^1 \, \ldots . \, x_n^1$$

and

$$X_2 = x_1^2 \; x_2^2 \; \ldots . \; x_n^2$$

be two vertices of the n-cube. We say

$$X_1 \leq X_2 \quad \text{iff} \quad x_i^1 \leq x_i^2 \quad (i=1,2,..,n) .$$

Definition:

A boolean function is said to be positive iff

$$X_1 \leq X_2 \rightarrow F(X_1) \leq F(X_2) .$$

With this definition, let X_1 be such that $F(X_1) = 1$ and where some of the x_i's are in negated form. Then there exists a subset X_k of vertices among the 2^n vertices such that $X_1 \leq X_k$ and by definition of positive function $F(X_k) = 1$.

Let us now go back to the n-cube. From all true vertices we take only those who are adjacent to some false vertex, and all them true boundary points. In similar fashion, we define the false boundary points. If we join the true boundary points with straight lines lying on the surface of the n-cube, we see that the n-cube surface has been divided into two disjoint surfaces, one containing the true vertices and the other the false vertices.

Example 3.1:

$F = x_1+x_2+x_3$; the dotted line in Fig. 3.1 is the line joining the boundary points.

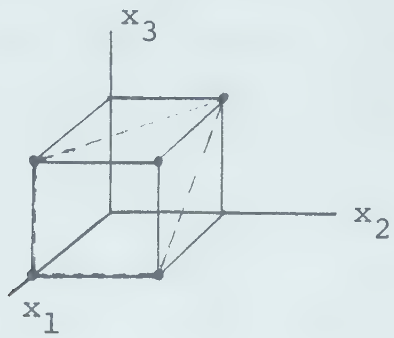


Fig. 3.1 Boundary line for $F = x_1+x_2x_3$

3.1 Separability of boolean functions

In this section we will prove that all unate functions are quadratically separable. But before going any further, we need the following definitions.

Assumability:

Let $F(x_1,x_2,..,x_n)$ be a boolean function of n variables. We say that F is k-summable, $k \geq 2$ iff for any integer $2 \leq j \leq k$ there are j true vertices $(X_1,X_2,..,X_j)$ not necessarily distinct and j false vertices $(Y_1,Y_2,..,Y_j)$ not necessarily distinct such that

$$\sum_{i=1}^{j} X_i = \sum_{i=1}^{j} Y_i \qquad F(X_i) = 1 \ , \quad F(Y_i) = 0 \ .$$

We say that F is assumable iff it is not summable for any $k \geq 2$.

<u>The space E^m:</u>

Let $F(x_1,x_2,..,x_n)$ be a boolean function of n variables in $B^n \varepsilon E^n$. We now define a new base E^m $(m=n(n+1)/2)$. There are at most m linearly independent vectors in E^m, and we choose these base vectors to be

$$y_1 = x_1$$
$$y_2 = x_2$$
$$.......$$
$$y_n = x_n$$
$$y_{n+1} = x_2.x_1$$
$$y_{n+2} = x_3.x_1$$
$$............$$
$$y_m = x_n.x_{n-1} \qquad m = n(n+1)/2$$

We have then transformed the function $F(x_1,x_2,..,x_n)$ into $F(y_1,y_2,..,y_n,....,y_m)$. If $F(y_1,y_2,..,y_m)$ is linearly separable in $B^m \varepsilon E^m$, that implies the existence of m+1 real parameters $b_1,b_2,..,b_m$ and T such that

$$b_1y_1+b_2y_2+...+b_ny_n+.....+b_my_m \geq T \quad \text{iff} \quad F(Y) = 1$$

(I)

$$b_1y_1+b_2y_2+...+b_ny_n+.....+b_my_m < T \quad \text{iff} \quad F(Y) = 0$$

but (I) is equivalent to

$$b_1x_1+b_2x_2+...+b_nx_n+.....+b_mx_n.x_{n-1} \geq T \quad \text{iff} \quad F(X) = 1$$

$$b_1x_1+b_2x_2+...+b_nx_n+.....+b_mx_n.x_{n-1} < T \quad \text{iff} \quad F(X) = 0$$

which is the condition to hold iff $F(X)$ is quadratic separable.

Theorem 3.1:

Let $F(x_1,x_2,..,x_n)$ be a boolean function of n variables. Then $F(y_1,y_2,..,y_m)$ is assumable in E^m iff $F(x_1,x_2,..,x_n)$ is quadratic. The proof is straightforward and it is therefore omitted.

Theorem 3.2 (Hruz):

If for a boolean function it is true that

$$\sum_{i=1}^{m} X_i = \sum_{i=1}^{m} Y_i \qquad F(X_i) = 1 \ , \quad F(Y_i) = 0 \ , \qquad m > 2$$

then there exist X_i', Y_i' $(i=1,2)$ such that

$$\sum_{i=1}^{2} X_i' = \sum_{i=1}^{2} Y_i' \qquad F(X_i') = 1 \ , \quad F(Y_i') = 0$$

that is, the given function is two-summable. The proof can be found in Hrunz [10].

Theorem 3.3

All unate functions are quadratic separable.

Proof:

Let $F(x_1,x_2,..,x_n)$ be a positive boolean function of n-variables. By theorem 3.1 we need only show that $F(y_1,y_2,..,y_m)$ is assumable in E^m. Assume the contrary; that is, suppose that for some k, F is k-summable in E^m. Then by theorem 3.2, there exist vectors X_i, Y_i $(i=1,2)$,

in E^n $F(X_i) = 1$, $F(Y_i) = 0$ such that

$$(*) \qquad \sum_{i=1}^{2} X_i^{(j)} . X_i^{(\ell)} = \sum_{i=1}^{2} Y_i^{(j)} . Y_i^{(\ell)} \qquad i \le j,\ell \le n$$

where $X_i^{(j)}$, $Y_i^{(j)}$ are the jth component of X_i and Y_i respectively. The theorem is proved by establishing the following contradiction: if equation (*) is satisfied for all $j = k$, that is, if

$$(**) \qquad \sum_{i=1}^{2} X_i^{(j)} = \sum_{i=1}^{2} Y_i^{(j)} \qquad 1 \le j \le n$$

then (*) cannot be satisfied for all $i \ne j$. Equation (**) takes the form

$$X_i^1 + X_i^2 = Y_i^1 + Y_i^2 \qquad (i=1,n) .$$

Let us order the coordinates and the vectors in such a way that

$$X_i^{(1)} = X_i^{(2)} = Y_i^{(1)} = Y_i^{(2)} = 0 \qquad i=1,p$$

$$X_i^{(1)} = X_i^{(2)} = Y_i^{(1)} = Y_i^{(2)} = 1 \qquad i=p+1,q$$

$$X_i^{(1)} = 1 \ , \quad X_i^{(2)} = 0 \qquad i=q+1,r$$

$$X_i^{(1)} = 0 \ , \quad X_i^{(2)} = 1 \qquad i=r+1,n$$

$$Y_i^{(1)} = 1 \ , \quad Y_i^{(2)} = 0 \qquad i=q+1,s$$

$$Y_i^{(1)} = 0 \quad , \quad Y_i^{(2)} = 1 \qquad i=s+1,t \ .$$

Obviously, $q < n$ because otherwise we would have $X^{(1)} = Y^{(1)}$ a contradiction. Two cases may occur

a.- $r \leq s$ then $X^{(1)} \leq Y^{(1)}$ and $F(X^{(1)}) > F(Y^{(1)})$

a contradiction since by definition of positive function $X \leq Y \rightarrow F(X) \leq F(Y)$.

b.- $r > s$ then $X_s = Y_s = (1\ 0)$

and $X_{s+1} = (1\ 0)$, $Y_{s+1} = (0\ 1)$

then $X_s^T . X_{s+1} = 1 \neq Y_s^T . Y_{s+1} = 0$ a contradiction to equation (*). We arrived to the conclusion that any positive function is assumable in E^m and therefore quadratic, and by Theorem 2.1 any unate function is quadratic realizable.

3.2 <u>Weights assigned to variables</u>

It has been shown that all unate functions are quadratic-realizable and in Chapter IV a method is given to find that realization. In this section we develop a method to assign a <u>weight</u> to each variable x_i $(i=1,n)$ and to the combination of two variables $x_i x_j$ $(i,j=1,n;\ i \neq j)$.

<u>The weight matrix</u>

We have already defined the generalized Chow parameters for a boolean function of n variables

$$c_i = c_{ii} = m(F_{x_i}) \qquad (i=1,n)$$

$$c_{ij} = m(F_{x_i x_j}) \qquad (j=1,n-1;\ i=j+1,n)$$

$$M = m(F)$$

where m is an operator such that if G is any given boolean function m(G) will give the number of true vertices of G.

For linear threshold functions it has been shown (Sheng, Lewis and Coates) that $c_i > c_j \rightarrow a_i > a_j$. For quadratic functions that is not true, in general. Since $0 \leq c_i \leq 2^{n-1}$ and $0 \leq c_{ij} \leq 2^{n-2}$, the upper bound on the c_i's is twice the upper bound on the c_{ij}'s, and this means that while c_i will usually be greater than any c_{ij}, it is often found necessary to assign to a given a_{ij} a greater weight than to a_i.

We have found that we can overcome this difficulty if we define the <u>relative weights</u> w_i, w_{ij} by the expression

$$w_i = w_{ii} = c_i/\min(M, 2^{n-1}) \qquad (i=1,n)$$

$$w_{ij} = c_{ij}/\min(c_i, c_j, 2^{n-2}) \qquad (j=1,n-1;\ i=j+1,n)$$

$$w_{ij} = 0 \qquad (j > i)$$

and the values of w_i, w_{ij} are now between 0 and 1. We are now in conditions to define the weight matrix

$$W = \left(\begin{array}{ccc|c} w_1 & 0 & 0 & 0 \\ w_{21} & w_2 & 0 & 0 \\ w_{31} & w_{32} & w_3 & 0 \\ \hline w_{n1} & w_{n2} & & w_n \end{array} \right)$$

Let $F(x_1,x_2,..,x_n)$ be a unate boolean function. The following theorems show the relation between its wirh weight matrix W and the realization matrix A.

Theorem 3.2.1:

If $w_i > \frac{1}{2}$, then

$$\sum_{k=1}^{i-1} a_{ik} + \sum_{k=i+1}^{n} a_{ki} > 0$$

and $a_i > 0$ is admissible.

Proof:

$$w_i > \frac{1}{2} \rightarrow c_i > M/2 \quad \text{iff} \quad M < 2^{n-1}$$

and

$$c_i > 2^{n-2} \quad \text{iff} \quad M > 2^{n-1}$$

and in either case, that implies F is positive in x_i and therefore $F_{x_i} \supset F_{\bar{x}_i}$. That means there exist two vertices Y,Z such that:

$$Y = x_1^* \; x_2^* \; \ldots \; x_i = 1 \; \ldots \; x_n^* \quad \text{and} \quad F(Y) = 1$$

$$Z = x_1^* \; x_2^* \; \ldots \; x_i = 0 \; \ldots \; x_n^* \quad \text{and} \quad F(Z) = 0$$

and since F is quadratic

$$Y^TAY = SUM + a_i + \sum_{k=1}^{i-1} a_{ik}x_k^* + \sum_{k=i+1}^{n} a_{ki}x_k^* \geq T$$

$$Z^TAZ = SUM + 0 \qquad < T$$

where SUM does not depend on a_i, a_{ik}. Looking at the last two inequalities, we get

$$a_i + \sum_{k=1}^{i-1} a_{ik}x_k^* + \sum_{k=i+1}^{n} a_{ki}x_k^* > 0$$

and since $x_k^* \geq 0$, it follows that

$$a_i > 0 \ , \quad \sum_{k=1}^{i-1} a_{ik} + \sum_{k=i+1}^{m} a_{ki} > 0$$

is admissible.

Theorem 3.2.2:

If $w_i < \frac{1}{2}$, then

$$\sum_{k=1}^{i-1} a_{ik} + \sum_{k=i+1}^{n} a_{ki} < 0$$

and $a_i < 0$ is admissible. The proof is analogous to Theorem 3.2.1 and is omitted.

Theorem 3.2.3:

If $w_i = \frac{1}{2}$, then $a_{ik} = 0$ $(k=1,i)$, $a_{ki} = 0$ $(k=i+1,n)$ is admissible.

Proof:

$$w_i = \frac{1}{2} \rightarrow c_i = M/2 \quad \text{iff} \quad M < 2^{n-1} \quad \text{or}$$

$$c_i = 2^{n-1} \quad \text{iff} \quad M > 2^{n-1}$$

and in either case, that implies $F_{x_i} = F_{\bar{x}_i}$ and that means there exist two points Y,Z such that

$$Y = x_1^* \; x_2^* \; \ldots \; x_i = 1 \; \ldots \; x_n^* \quad \text{and} \quad F(Y) = 1$$

$$Z = x_1^* \; x_2^* \; \ldots \; x_i = 0 \; \ldots \; x_n^* \quad \text{and} \quad F(Z) = 0$$

Since both are true vertices, then

$$Y^T AY = SUM + a_i + \sum_{k=1}^{i-1} a_{ik}x_k^* + \sum_{k=i+1}^{n} a_{ki}x_k^* \geq T$$

$$Y^T AZ = SUM + \quad 0 \quad < T$$

We have that

$$a_i + \sum_{k=1}^{i-1} a_{ik}x_k^* + \sum_{k=i+1}^{n} a_{ki}x_k^* = 0$$

is an acceptable solution, and since $x_k^* \geq 0$, we conclude that $a_{ik} = 0$ $(k=1,i)$, $a_{ki} = 0$ $(k=i+1,n)$ is an admissible solution.

<u>Theorem 3.2.4</u>:

$w_i > w_j$ implies

$$\sum_{k=1}^{i-1} a_{ik} + \sum_{k=i+1}^{n} a_{ki} > \sum_{m=1}^{j-1} a_{jm} + \sum_{m=j+1}^{n} a_{mj}$$

and $a_i > a_j$ is admissible.

Proof:

We can prove the theorem in two parts, first for the special case when F is 2-monotonic, and then, for unate functions in general.

Case 1:

F is 2-monotonic. Then $w_i > w_j \rightarrow c_i > c_j$ and this in turn implies (Sheng) $F_{x_i\bar{x}_j} \supset F_{\bar{x}_i x_j}$ and $F_{\bar{x}_i x_j} \not\subset F_{x_i\bar{x}_j}$. Then there exist two vertices Y,Z such that

$$Y = x_1^* \; x_2^* \ldots x_i = 1 \ldots x_j = 0 \ldots x_n^* \quad \text{and} \quad F(Y) = 1$$

$$Z = x_1^* \; x_2^* \ldots x_i = 0 \ldots x_j = 1 \ldots x_n^* \quad \text{and} \quad F(Z) = 0 \; .$$

And since F is quadratic

$$Y^T AY = SUM + a_i + \sum_{\substack{k=1 \\ k=j}}^{i-1} a_{ik} x_k^* + \sum_{\substack{k=i+1 \\ k \neq j}}^{n} a_{ki} x_k^* \geq T$$

$$Z^T AZ = SUM + a_j + \sum_{\substack{m=1 \\ m \neq i}}^{j-1} a_{jm} x_m^* + \sum_{\substack{m=j+1 \\ m \neq i}}^{n} a_{mj} x_m^* < T$$

where SUM does not depend on either a_{ik} or a_{jm}. From the last two equations we get

$$a_i + \sum_{\substack{k=1 \\ k \neq j}}^{i-1} a_{ik} x_k^* + \sum_{\substack{k=i+1 \\ k \neq j}}^{n} a_{ki} x_k^* > a_j + \sum_{\substack{m=1 \\ m \neq i}}^{j-1} a_{jm} x_m^* + \sum_{\substack{m=i+1 \\ m \neq j}}^{n} a_{mj} x_m^* \; .$$

Adding a_{ij} to both sides of the equation and setting

$$A_i = (a_{i1}\ a_{i2} \ldots a_i \ldots a_{ni});$$

$$A_j = (a_{j1}\ a_{j2} \ldots a_j \ldots a_{nj})$$

and

$$X^* = (x_1^*\ x_2^*\ \ldots x_i = 1 \ldots x_j = 1 \ldots x_n^*)$$

we have

$$X^{*T}A_i > X^{*T}A_j$$

and since $(X^*)_i \geq 0$, follows that $|A_i| > |A_j|$ is admissible, and hence $a_i > a_j$,

$$\sum_{k=1}^{i-1} a_{ik} + \sum_{k=i+1}^{n} a_{ki} > \sum_{m=1}^{j-1} a_{jm} + \sum_{m=i+1}^{n} a_{mj}$$

is admissible.

Case 2:

Suppose F is not 2-monotonic. Although $F_{x_i \bar{x}_j}$ and $F_{\bar{x}_i x_j}$ can no longer be compared (Sheng), there still must exist two vertices Y,Z such that

$$Y = x_1^*\ x_2^*\ \ldots x_i = 1 \ldots x_j = 0 \ldots x_n^* \quad \text{and} \quad F(Y) = 1$$

$$Z = x_1^*\ x_2^*\ \ldots x_i = 0 \ldots x_j = 1 \ldots x_n^* \quad \text{and} \quad F(Z) = 0$$

Necessity:

Let $Y_k = x_{1k}^*\ x_{2k}^* \ldots x_i = 1 \ldots x_j = 0 \ldots x_{nk}^*$ be a true vertex and let $Z_k = x_{1k}^*\ x_{2k}^* \ldots x_i = 0 \ldots x_j = 1 \ldots x_{nk}^*$ be another vertex. Let K_i be the number of true vertices with $x_i = 1$,

$x_j = 0$. Then, $F(Y_k) = 1$ $(k=1,K_i)$. Let us assume $F(Z_k) = 1$ $(k=1,K_i)$. But that would imply x_j contains at least as many vertices as x_i, or $c_i \le c_j$, contrary to our assumption. On the contrary, if $F(Z_k) = 0$ for some k, then there exist two vertices Y and Z as defined above. Using the same reasoning as in case 1, we conclude

$$w_i > w_j \rightarrow \sum_{k=1}^{i-1} a_{ik} + \sum_{k=i+1}^{n} a_{ki} > \sum_{m=1}^{j-1} a_{jm} + \sum_{m=j+1}^{n} a_{mj}$$

and

$$a_i > a_j$$

is admissible.

<u>Theorem 3.2.5</u>:

If $w_{ik} \ge w_{jm}$, then $a_{ik} \ge a_{jm}$ $(i,j,k,m=1,n)$ is admissible.

<u>Proof</u>:

$w_i > w_j \rightarrow w_{ik} > w_{jk}$ and this is obvious since the contribution of the coordinate <u>i</u> must be greater than or equal to the contribution of coordinate <u>j</u>. We then have:

Contribution of coord.<u>i</u>: $w_{i1}\ w_{i2} \cdot\cdot w_i \cdot\cdot\cdot\cdot\cdot w_{ni}$

Contribution of coord.<u>j</u>: $w_{j1}\ w_{j2} \cdot\cdot\cdot\cdot\cdot\cdot w_j \cdot\cdot w_{nj}$

If we accept $w_{ik} \ge w_{jm} \rightarrow a_{ik} \ge a_{jm}$, then

$$\sum_{k=1}^{i} a_{ik} + \sum_{k=i+1}^{n} a_{ki} > \sum_{m=1}^{j} a_{jm} + \sum_{m=j+1}^{n} a_{mj}$$

if $w_i > w_j$ and this result is admissible, according to Theorem 3.2.4.

Theorem 3.2.6:

If $w_i = 1$ then $a_i \geq T$.

Proof:

$w_i = 1$ implies that all points with $x_i = 1$ are true vertices and in particular $X_0 = 0\ 0\ \ldots.\ x_i = 1\ldots.0$ is a true vertex. Then $X_0^T A X_0 = a_i \geq T$.

Theorem 3.2.7:

If $w_{ij} = 1$ then $a_i + a_j + a_{ij} \geq T$.

Proof:

$w_{ij} = 1$ implies that all points with $x_i = x_j = 1$ are true vertices and in particular $X_0 = 0\ 0\ \ldots x_i = 1\ldots x_j = 1..0$ is a true vertex. Then $X_0^T A X_0 = a_i + a_j + a_{ij} \geq T$.

Theorem 3.2.8:

Let $F = F(x_1, x_2, .., x_n)$ be a positive boolean function. A realization $[A;T]$ such that $|A|_2 = \min$ is one such that A is positive definite, and $[A;T]$ is called a minimal realization.

Proof:

If $[A;T]$ is a realization for F, then

$$G_i = X_i^T A X_i - T \geq 0 \qquad (i=1,M)$$

and

$$G_j = X_j A X_j - T < 0 \qquad (j=M,2^n)$$

For each point $X_j \varepsilon \bar{F}$ we take another $X_i \varepsilon F$ such that $X_i \supset X_j$ and form

$$L_k(A) = G_i - G_j = X_i^T A X_i - T - X_j^T A X_j + T = a_u + a_w + \ldots + a_{xz} > 0$$

where $a_u, a_w, \ldots, a_{xz}$ are coefficients present in G_i but not in G_j. Similarly, for each point $X_m \varepsilon F$ we take another $X_n \varepsilon \bar{F}$ such that $X_m \supset X_n$ and form

$$L_h(A) = G_m - G_n = X_m^T A X_m - T - X_n^T A X_n + T = a_b + a_c + \ldots + a_{de} > 0 \ .$$

We have then 2^n equations, not all necessarily different

$$L_i(A) = \text{Subset}_i(A) = S_i(A) > 0 \qquad (i=1,2^n) \ .$$

Our problem becomes

$$f(A) = |A|_2 = |a_1| + |a_2| + \ldots + |a_n| + \ldots + |a_{ij}| += \min$$

subject to $L_i(A) = S_i(A) > 0$.

The solution (not unique) for this problem is one such that $a_i \geq 0$ $(i=1,n)$, $a_{ij} \geq 0$ $(j=1,n-1;i=j+1,n)$ but that means that A is positive definite.

CHAPTER IV : REALIZATION OF UNATE QUADRATIC FUNCTIONS

As we have seen in Chapter III, any unate function is quadratically separable. Furthermore, there are some non-unate functions for which a quadratic realization is possible, although the realization process is more ardous than for unate function and no general method has yet been developed.

4.1 Positive realization

Let $F(x_1,x_2,..,x_n)$ be a unate function. From Theorem 2.1, the realizations of $F(x_1,x_2,..,x_i,..,x_n)$ and $F(x_1,x_2,..,\bar{x}_i,..,x_n)$ are related to each other and, therefore, given a non-positive unate function, we can find the realization corresponding to the positive function resulting from complementing the original negated variables, and then go back to the realization for the original function using the relations from Theorem 2.1.

From Theorem 3.2.8, we know that for a positive function all the coefficients a_{ij} will be positive and it is then very simple to find the maximum and minimum set, as explained below.

Definition:

Let $F(x_1,x_2,..,x_n)$ be a positive boolean function. The minimum set X_M is the set of vertices belonging to

the boundary of F, or true boundary vertices. The <u>maximum set</u> X_m is the set of vertices belonging to the boundary of $\bar{F}$, or false boundary vertices.

<u>Example 4.1</u>

For $F_1 = x_1x_2+x_1x_3+x_2x_3$ the minimum and maximum sets are, respectively:

$X_M = [1\ 1\ 0,\ 1\ 0\ 1,\ 0\ 1\ 1]$

$X_m = [1\ 0\ 0,\ 0\ 1\ 0,\ 0\ 0\ 1]$

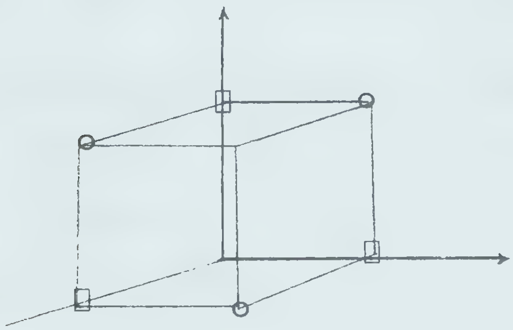


Fig 4.1 Min o and
max □ sets for F_2

<u>Theorem 4.1</u>:

Let $F(x_1,x_2,..,x_n)$ be a positive boolean function. If R[F] = [A;T] is a realization for the min and max sets, then R[F] realizes F.

<u>Proof</u>:

Let U be a vertex of X_M, V a vertex of X_m,

then $U^TAU \geq T$ and $V^TAV < T$.

If X_K is any true vertex, then there exists some true vertex $X_u \varepsilon X_M$ such that $X_u \leq X_k$ and then

$$T \leq X_u^T A X_u \leq X_k^T A X_k \quad .$$

Similarly, if X_h is a false vertex, then there exists some false vertex $X_v \varepsilon X_m$ such that $X_v \geq X_h$ and

$$T > X_v^T A X_v \geq X_h^T A X_h \quad .$$

We conclude then that [A;T] is a realization for F.

<u>Realization method for positive functions</u>

Let $F(x_1,x_2,..,x_n)$ be a positive boolean function. To find its quadratic realization is to solve for [A;T] the 2^n equations:

$$X^T AX \geq T \qquad \text{iff} \qquad F(X) = 1 \quad (M \text{ equations})$$

and

$$X^T AX < T \qquad \text{iff} \qquad F(X) = 0 \quad (2^n\text{-}M \text{ equations})$$

We have seen, however, that the problem is greatly simplified if we find the maximum and minimum set, since it reduces the number of vertices under consideration from 2^n to (p+q), where (p,q) denotes the number of vertices in (X_M,X_m). Finding the realization of a given function is then to solve the system of (p+q) equations

$$a_1x_1^1+a_2x_2^1+\ldots\ldots+a_nx_n^1+\ldots\ +a_{ij}x_i^1x_j^1+\ \ldots \geq T$$

$$a_1x_1^2+a_2x_2^2+\ldots\ldots+a_nx_n^2+\ldots\ +a_{ij}x_i^2x_j^2+\ \ldots \geq T$$

$$\ldots\ldots\ldots\ldots$$

$$a_1x_1^p+a_2x_2^p+\ldots\ldots+a_nx_n^p+\ldots\ +a_{ij}x_i^px_j^p+\ \ldots \geq T$$

$$a_1x_1^{p+1}+a_2x_2^{p+1}+\ldots+a_nx_n^{p+1}+\ldots+a_{ij}x_i^{p+1}x_j^{p+1}+\ldots < T$$

$$\ldots\ldots\ldots\ldots$$

$$a_1x_1^{p+q}+a_2x_2^{p+q}+\ldots+a_na_n^{p+q}+\ldots+a_{ij}x_i^{p+q}x_j^{p+q}+\ldots < T$$

For the $1+n(n+1)/2$ parameters $a_1,a_2,\ldots,a_n,a_{21},\ldots,a_{n,n-1}$ and T.

There are many ways to solve the above system. One might think that the most direct approach would be the simplex method, since its implementation in a computer program is straightforward; however, the size of the matrix grows too rapidly with n $(n(n+1)/2)$ and it is impractical when dealing with more than 5 or 6 variables.

We have developed an algorithm, based on Sheng's successive higher ordering and which provides satisfactory results.

4.2 Realization of a unate boolean function by successive higher ordering method

Preliminaries: The successive higher ordering method is a method which gives minimal quadratic realization for

any unate boolean function. Let $F(x_1,x_2,..,x_n)$ be a unate positive boolean function and let $w_1,w_2,..,w_n,$ $w_{21},..,w_{n,n-1}$ be its relative weights. For reasons of greater clarity, we will number them as $w_1,w_2,..,w_n,w_{n+1},$ $w_{n+2},...,w_m$ with $m = (n(n+1)/2$. Since F is positive, it is possible to find a realization such that all a_{ij}'s are positive numbers.

Let us assume the initial ordering among them is such that $w_1 \geq w_2 \geq w_3 \geq .... \geq w_m$. We have shown before that $w_i \geq w_j$ implies $a_i \geq a_j$ is admissible. Then $w_1 \geq w_2 \geq w_3 \geq ... \geq w_m \rightarrow a_1 \geq a_2 \geq a_3 \geq ... \geq a_m$. If we take $w_i = w_j \rightarrow a_i = a_j$ we are left then only with the strict inequalities

$$a_1 > a_i > a_k > ... > a_p > 0$$

and we have reduced the number of unknowns from m to m' = m-E, where E is the number of equal signs.

There is still another reduction we can accomplish. Realizing a function means solving the equations

$$X_k^T A X_k \geq T \qquad (k=1,2,..,p)$$

and

$$X_j^T A X_j < T \qquad (j=1,2,..,q) \tag{4.1}$$

where p and q are the number of vertices in the maximum and minimum set. Let us now assume we have equalities of the type

$$\sum_{k=1}^{n} w_{ik} = \sum_{k=1}^{n} w_{jk} \quad .$$

That means that the contribution of the coordinate x_i is the same of coordinate x_j; we denote that by saying $x_i = x_j$ and can consequently reduce the number of equations from $(p+q)$ to $(p'+q')$.

Example 4.2.

$F_1 = x_1x_2+x_3x_4$. The Chow matrix is

$$C = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 4 & 5 & 0 & 0 \\ 3 & 3 & 5 & 0 \\ 3 & 3 & 4 & 5 \end{pmatrix} \qquad M = 7$$

and the weight matrix's elements are $w_i = w_{ii} = c_i/\min(M,2^{n-1})$ $(i=1,n)$ and $w_{ij} = c_{ij}/\min(M,c_i,c_j)$ $(j=1,n-1;i=j+1,n)$. We have then

$$\sum_{k=1}^{n} w_{1k} = w_{11}+w_{21}+w_{31}+w_{41} = 5/7+4/4+3/4+3/4$$

$$\sum_{k=1}^{n} w_{2k} = w_{21}+w_{22}+w_{32}+w_{42} = 4/4+5/7+3/4+3/4$$

$$\sum_{k=1}^{n} w_{3k} = w_{31}+w_{32}+w_{33}+w_{43} = 3/4+3/4+5/7+4/4$$

$$\sum_{k=1}^{n} w_{4k} = w_{41}+w_{42}+w_{43}+w_{44} = 3/4+3/4+4/4+5/7$$

We conclude that $x_1 = x_2 = x_3 = x_4$ and the minimum set can then be reduced from $X_M = (1\ 1\ 0\ 0,\ 0\ 0\ 1\ 1)$ to $X_{M'} = (1\ 1\ 0\ 0)$, the maximum set from $X_m = (1\ 0\ 0\ 1,\ 1\ 0\ 1\ 0,\ 0\ 1\ 1\ 0,\ 0\ 1\ 0\ 1)$ to $X_{m'} = (1\ 0\ 0\ 1)$ and the number of equation under consideration has been reduced from $p+q = 2+4 = 6$ to $p'+q' = 1+1 = 2$.

The solution will be such that $a_1 = a_2 = a_3 = a_4$; $a_{12} = a_{34}$; $a_{13} = a_{14} = a_{23} = a_{24}$.

4.3 General method

Let $F(x_1, x_2, .., x_n)$ be a positive boolean function and assume that all its weights are different and in the following order (we will justify later why we can assume them to be all different)

$$w_1 > w_2 > w_3 > \ldots > w_m \quad , \tag{4.2}$$

that means

$$a_1 > a_2 > a_3 > \ldots > a_m > 0 \quad .$$

Equation (4.3) can be written as

$$\begin{aligned} a_m &= a_m \\ a_{m-1} &= a_m + \Delta_{m-1} \\ a_{m-2} &= a_m + \Delta_{m-1} + \Delta_{m-2} \\ \ldots & \quad \ldots\ldots \end{aligned} \tag{4.4}$$

$$a_2 = a_m + \Delta_{m-1} + \quad_{m-2} + \cdots\cdots + \Delta_2$$

$$a_1 = a_m + \Delta_{m-1} + \Delta_{m-2} + \cdots\cdots + \Delta_2 + \Delta_1$$

where $\Delta_i \geq 0$ $(i=1,2,..,m-1)$. If $w_{m-i} = w_{m-i-1}$, then $\Delta_{m-i-1} = 0$ and our assumption of considering all weights to be different is then valid.

We have seen then that X_M contains p vertices which can be reduced to p' ($p \geq p'$) should the equality

$$\sum_{k=1}^{n} w_{ik} = \sum_{k=1}^{n} w_{jk}$$

hold for any pair of i and j. In the same fashion, X_m contains q vertices which can be reduced to q' and Equation (4.1) can be written

$$\begin{aligned} X_j^T A X_j &\geq T \qquad (j=1,p') \\ X_k^T A X_k &< T \qquad (k=1,q') \quad . \end{aligned} \tag{4.5}$$

Equation (4.5) can also be written

$$L_j(A) > R_k(A) \qquad (j=1,p';k=1,q') \tag{4.6}$$

and we have transformed (4.5) a system of p'+q' equations with m+1 unknowns $(a_1,a_2,..,a_m,T)$ into a system of p'+q' equations with m unknowns $(a_1,a_2,..,a_m)$.

Using equations (4.4) system (4.6) is transformed into a system where then unknowns are $\Delta_1,\Delta_2,..,\Delta_{m-1},a_m$,

that is

$$L_j(\Delta,a_m) > R_k(\Delta,a_m) \quad (j-1,p';k=1,q') \qquad (4.7)$$

Example 4.3:

$$F_1 = x_1x_2+x_3x_4 \ .$$

We know from Example 4.2

$$w_{21} = w_{43};\ w_1-w_2=w_3=w_4,\ w_{31}=w_{41}=w_{43}$$

which in turn implies

$$a_{21} = a_{43};\ a_1=a_2=a_3=a_4,\ a_{31}=a_{41}=a_{43}$$

or using the increments Δ_is

$$a_{31} = a_{41} = a_{43} \quad = a_{31}$$

$$a_1 = a_2 = a_3 = a_4 = a_{31} + \Delta_1$$

$$a_{21} = a_{43} \qquad = a_{31} + \Delta_1 + \Delta_{21}$$

Since $X_{M'} = (1\ 1\ 0\ 0)$ and $X_{m'} = (1\ 0\ 0\ 1)$ we have

$$L_1 = X_{M1}AX_{M1} = a_1+a_2+a_{21} = 3a_{31} + 3\Delta_1 + \Delta_{21}$$

$$R_1 = X_{m1}AX_{m1} = a_1+a_4+a_{41} = 3a_{31} + 2\Delta_1$$

then $L_1 > R_1$ implies

$$3a_{31} + 3\Delta_1 + \Delta_{21} > 3a_{31} + 2\Delta_1$$

or

$$\Delta_1 + \Delta_{21} > 0 \ .$$

One solution is $a_{31} = \Delta_1 = 0$; $\Delta_{21} = 1$, which gives

$$a_{31} = a_{41} = a_{32} = 0$$

$$a_1 = a_2 = a_3 = a_4 = a_{31} + \Delta_1 = 0+0 = 0$$

$$a_{21} = a_{43} = a_{31} + \Delta_1 + \Delta_{21} = 0+0+1 = 1 \quad .$$

The matrix A is then

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and $T = L_1(A) = 1$.

Note that $\Delta_1 + \Delta_{21} > 0$ can have infinite solutions, but they must satisfy the condition $a_{21} > a_1$, so $\Delta_1 = 1$; $\Delta_{21} = 0$ is not acceptable, because although it would satisfy $\Delta_1 + \Delta_{21} > 0$, it would also imply $a_1 > a_{21}$. Also the solution does not depend on a_{31} and since we are looking for a solution which minimizes norm(A) $= \sum_{k=1}^{n} |a_{ik}|$ $(i=1,n)$, we choose $a_{31} = 0$.

General procedure

The following is the general procedure when we want to find the quadratic realization for a unate boolean function.

a) Check the function for unateness and change it, when necessary, to a positive boolean function.

b) Find the complementary function $\bar{F}(x_1,x_2,..,x_n)$.

c) Determine X_M and X_m, minimum and maximum set, and p and q.

d) Find the weight matrix and the initial ordering

$$w_1 \geq w_2 \geq w_3 \geq \dots \geq w_m$$

e) Find i and j in X_M such that

$$\sum_{k=1}^{n} w_{ik} = \sum_{k=1}^{n} w_{jk}$$

and i and j in X_m such that

$$\sum_{k=1}^{n} w_{ik} = \sum_{k=1}^{n} w_{jk}$$

and reducing in this way p and q to p' and q'.

f) Form the equations

$$L_j(A) \geq T \qquad (j=1,p')$$

$$R_k(A) < T \qquad (k=1,q')$$

g) Substitution of a_i's for Δ_i's $(i=1,m-1)$.

h) Express the p'+q' equations

$$L_j(a_m,\Delta) \geq T \qquad (j=1,p')$$

$$R_k(a_m,\Delta) < T \qquad (k=1,q')$$

into the form $L_j(a_m,\Delta) > R_k(a_m,\Delta)$ $(j=1,p';k=1,q')$.

i) Check for $L_j > R_k$.

I. If $L_j > R_k$ for all j and k, then the function is in quadratic form and we can go on to step j, but if

II. $L_j \leq R_k$ for some j and k, then of the equations $L_j \leq R_k$, simplifying common terms, we will get inequalities of two types:

Type 1: of the kind $\Delta_i > \sum_k m_k \Delta_k$, where k is an integer number (one Δ_i on the left hand side, two or more Δ_k on the right hand side).

Type 2: of the kind $\sum_i m_i \Delta_i > \sum_j m_j \Delta_j$, where m_i, m_j are integers, two or more Δ_i on both sides.

For inequalities of Type 1, we set

$$\Delta_i = \Delta_i^2 + \sum_k m_k \Delta_k$$

and substituting Δ_i by its value on $\Delta_i > \sum m_k \Delta_k$ we get

$$\Delta_i = \Delta_i^2 + \sum_k m_k \Delta_k > \sum_k m_k \Delta_k \quad \text{and} \quad \Delta_i^2 > 0$$

as expected, since we want all increments to be positive. Now replacing Δ_i by its new expression

$\Delta_i^2 + \sum_k m_k \Delta_k$ we add a positive increment to L_j, and the number of inequalities $L_j \leq R_k$ is bound to decrea decrease.

For inequalities of Type 2 we will have situations like

$$\Delta_1 + \Delta_2 > \Delta_3 + \Delta_4$$

and if this is true, at least one of the following inequalities must hold $\Delta_1 > \Delta_3$; $\Delta_2 > \Delta_3$; $\Delta_1 > \Delta_4$ or $\Delta_2 > \Delta_4$. We have no way to decide which one to choose and we must resort to trial and error. A rule of thumb says that we should try to pick one such that the one on the left (i.e. Δ_1) is more often in L_j that in R_k. Suppose we choose $\Delta_1 > \Delta_4$, then $\Delta_1 = \Delta_i^2 + \Delta_4$. The process has to be repeated until the inequality $L_j > R_k$ holds for all j and k.

j) Suppose we have already that $L_j > R_k$ holds for all j and k, and assume the initial ordering was

$$a_1 \geq a_2 \geq \ldots \geq a_m .$$

We then set $a_m = 0$ and if $L_j > R_k$ does not hold any more, we then set

$$a_m = \Delta_{m-1}^{i_{m-1}} = \Delta_{m-2}^{i_{m-2}} \ldots. = \Delta_1^{i_1} = 1$$

and go to step <u>k</u> but if $L_j > R_k$ still holds, we make

$$\Delta_{m-1}^{i_{m-1}} = 0$$

and check if $L_j > R_k$ holds for all j and k. If it does not, then we set

$$a_m = 0; \quad \Delta_{m-1}^{i_{m-1}} = \Delta_{m-2}^{i_{m-2}} = \ldots. = \Delta_1^{i_1} = 1$$

and go to step <u>k</u>. But if $L_j > R_k$ still holds then we set $\Delta_{m-2}^{i_{m-2}} = 0$ and ......

The procedure should be clear by now. All we are trying to do is to minimize norm(A) $= \sum_{k=1}^{n} |a_{ik}|$ (i=1,n) by setting to zero as many increments as we can.

k) We are now ready to find the threshold T and the matrix A. Since the increments are known, we have

$$a_m = a_m$$

$$a_{m-1} = a_m + \Delta_{m-1}$$

$$a_{m-2} = a_m + \Delta_{m-1} + \Delta_{m-2}$$

....

$$a_1 = a_m + \Delta_{m-1} + \Delta_{m-2} + \cdots\cdots + \Delta_1$$

and

$$T = \min_j (X_i^T A X_j) \qquad (j=1,p') .$$

ℓ) If the function $F(x_1,x_2,..,x_n)$ was positive to begin with, then [A;T] is the final realization, but if the original function F was not positive, then the realization will be [A';T'] where [A';T'] and [A;T] are related by Theorem 2.1.

<u>Example 4.4</u>:

$$F_1 = x_1x_2+x_1x_3+x_1x_4x_5x_6+x_2x_3x_4+x_2x_3x_5+x_2x_4x_5x_6.$$

We have $c_1 = 25$, $c_2 = 23$, $c_3 = 22$, $c_4 = 18$, $c_5 = 18$, $c_6 = 17$; $c_{21} = 16$, $c_{31} = 16$, $c_{41} = 13$, $c_{51} = 13$, $c_{61} = 13$; $c_{32} = 14$, $c_{42} = 13$, $c_{52} = 13$, $c_{62} = 12$, $c_{43} = 12$, $c_{53} = 12$, $c_{63} = 11$; $c_{54} = 10$, $c_{64} = 10$; $c_{65} = 10$.

We can now find the weight matrix elements, $w_1 = c_i/\min(2^{n-1},M)$, and $w_{ij} = c_{ij}/\min(2^{n-2},c_i,c_j)$. Since $M = 32$, $\min(2^{n-1},M) = 32$ and the matrix W has the components

$w_1 = 25/32 = 0.78$, $w_2 = 23/32 = 0.71$, $w_3 = 22/32 = 0.687$,
$w_4 = 18/32 = 0.562 = w_5$, $w_6 = 17/32 = 0.53$;

Since c_i $2^{n-2} = 16$ for i=1,6, we have that
$\min(2^{n-2},c_i,c_j) = 2^{n-2} = 16$ and then

$w_{21} = w_{31} = 16/16 = 1$; $w_{41} = w_{51} = w_{61} = w_{42} = w_{52} = 13/16 = 0.81$;
$w_{32} = 14/16 = 0.87$; $w_{62} = w_{43} = w_{53} = 12/16 = 0.75$;
$w_{63} = 11/16 = 0.68$; $w_{54} = w_{64} = w_{65} = 10/16 = 0.62$.

The initial ordering is then

$$w_{12} = w_{31} \quad w_{32} \quad w_{41} = w_{51} = w_{61} = w_{42} = w_{52} \quad w_1 \quad w_{62} = w_{43} = w_{53} \quad w_2$$
$$w_3 = w_{63} \quad w_{54} = w_{64} = w_{65} \quad w_4 = w_5 \quad w_6$$

Since equal weights implies equal coefficients, Δ_{21} designs both Δ_{21} and Δ_{31}, Δ_{41} designs Δ_{41}, Δ_{51}, Δ_{61}, Δ_{42} and Δ_{52} and so on.

The minimum and maximum sets are

$$X_M' = [0\ 1\ 0\ 1\ 1\ 1,\ 0\ 1\ 1\ 0\ 1\ 0,\ 1\ 0\ 0\ 1\ 1\ 1,\ 1\ 0\ 1\ 0\ 0\ 0,$$
$$1\ 1\ 0\ 0\ 0\ 0]$$

$$X_m' = [0\ 0\ 1\ 1\ 1\ 1,\ 0\ 1\ 0\ 0\ 1\ 1,\ 0\ 1\ 0\ 1\ 1\ 0,\ 0\ 1\ 1\ 0\ 0\ 1,$$
$$1\ 0\ 0\ 0\ 1\ 1,\ 1\ 0\ 0\ 1\ 1\ 0]\ .$$

We now write

$a_6 = a_6$

$a_4 = a_6 + \Delta_4$

$a_{54} = a_6 + \Delta_4 + \Delta_{54}$

$a_3 = a_6 + \Delta_4 + \Delta_{54} + \Delta_3$

$a_2 = a_6 + \Delta_4 + \Delta_{54} + \Delta_3 + \Delta_2$

$a_{62} = a_6 + \Delta_4 + \Delta_{54} + \Delta_3 + \Delta_2 + \Delta_{62}$

$a_1 = a_6 + \Delta_4 + \Delta_{54} + \Delta_3 + \Delta_2 + \Delta_{62} + \Delta_1$

$a_{41} = a_6 + \Delta_4 + \Delta_{54} + \Delta_3 + \Delta_2 + \Delta_{62} + \Delta_1 + \Delta_{41}$

$a_{32} = a_6 + \Delta_4 + \Delta_{54} + \Delta_3 + \Delta_2 + \Delta_{62} + \Delta_1 + \Delta_{41} + \Delta_{32}$

$a_{21} = a_6 + \Delta_4 + \Delta_{54} + \Delta_3 + \Delta_2 + \Delta_{62} + \Delta_1 + \Delta_{41} + \Delta_{32} + \Delta_{21}$

Designing the vertices of X_M, by I, II, III, IV and V and the vertices of X_m, by VI, VII, VIII, IX, X and XI we form the

Table I

	Δ_{21}	Δ_{32}	Δ_{41}	Δ_1	Δ_{62}	Δ_2	Δ_3	Δ_{54}	Δ_4	a_6
I	0	0	2	2	3	4	4	7	9	10
II	0	1	2	2	3	4	5	5	6	6
III	0	0	3	4	4	4	4	7	9	10
IV	1	1	1	2	2	2	3	3	3	3
V	1	1	1	2	2	3	3	3	3	3
VI	0	0	0	0	2	2	4	7	9	10
VII	0	0	1	1	2	3	3	4	5	6
VIII	0	0	2	2	2	3	3	4	6	6
IX	0	1	1	1	2	3	5	5	5	6
X	0	0	2	3	3	3	3	4	5	6
XI	0	0	2	3	3	3	3	4	6	6

In Table I, subtracting from each column the smallest element we get

Table II

	Δ_{21}	Δ_{32}	Δ_{41}	Δ_1	Δ_{62}	Δ_2	Δ_3	Δ_{54}	Δ_4	a_6	
I	0	0	2	2	1	2	1	4	6	7	25
II	0	1	2	2	1	2	2	2	3	3	18
III	0	0	3	4	2	2	1	4	6	7	29
IV	1	1	1	2	0	0	0	0	0	0	5
V	1	1	1	2	0	0	0	0	0	0	6
VI	0	0	0	0	0	0	1	4	6	7	18
VII	0	0	1	1	0	1	0	1	2	3	9
VIII	0	0	2	2	0	1	0	1	3	3	12
IX	0	1	1	1	0	1	2	2	2	3	13
X	0	0	2	3	1	1	0	1	2	3	13
XI	0	0	2	3	1	1	0	1	3	3	14

Comparing V and IX

$$\Delta_{12} + \Delta_{23} + \Delta_{41} + 2\Delta_1 + \Delta_2 > \Delta_{32} + \Delta_{41} + \Delta_1 + \Delta_2 + 2\Delta_3 + 2\Delta_{45} + 3a_6$$

or

$$\Delta_{21} + \Delta_1 > 2\Delta_3 + 2\Delta_{54} + 2a_6 \ .$$

And since Δ_{21} is only in L_j we can safely take

$$\Delta_{21} = \Delta_{21}^2 + 2\Delta_3 + 2\Delta_{54} + 3a_6 + 2\Delta_4 \quad .$$

We replace Δ_{21} by its new value and form

Table III

Vertex	Δ_{21}^2	Δ_{32}	Δ_{41}	Δ_1	Δ_{62}	Δ_2	Δ_3	Δ_{54}	Δ_4	a_6	SUM
I	0	0	2	2	1	2	1	4	6	7	25
II	0	1	2	2	1	2	2	2	3	3	18
III	0	0	3	4	2	2	1	4	6	7	29
IV	1	1	1	2	0	0	2	2	2	3	14
V	1	1	1	2	0	1	2	2	2	3	15
VI	0	0	0	0	0	0	1	4	6	7	18
VII	0	0	1	1	0	1	0	1	2	3	9
VIII	0	0	2	2	0	1	0	1	3	3	12
IX	0	1	1	1	0	1	2	2	2	3	13
X	0	0	2	3	1	1	0	1	2	3	13
XI	0	0	2	3	1	1	0	1	3	3	14

We see that it is not already in quadratic form, since IV = XI.

Comparing IV with XI we get

$$\Delta_{21}^{2} + \Delta_{32} + \Delta_{41} + 2\Delta_1 + 2\Delta_3 + 2\Delta_{54} + 2\Delta_4 + 3a_6$$

$$> 2\Delta_{41} + 3\Delta_1 + \Delta_{62} + \Delta_2 + \Delta_{54} + 2\Delta_4 + 3a_6$$

$$\Delta_{21}^{2} + 2\Delta_{32} + 2\Delta_3 + \Delta_{54} > \Delta_{41} + \Delta_1 + \Delta_{62} + \Delta_2 + \Delta_4$$

We take $\Delta_{21}^{2} > \Delta_2$, since Δ_{21} is only in L_j. Then

$$\Delta_{21}^{2} = \Delta_{21}^{3} + \Delta_2$$

Table IV

Vertex	21^3	32	41	1	62	2	3	54	4	6	SUM
I	0	0	2	2	1	2	1	4	6	7	25
II	0	1	2	2	1	2	2	2	3	3	18
III	0	0	3	4	2	2	1	4	6	7	29
IV	1	1	1	2	0	1	2	2	2	3	15
V	1	1	1	2	0	2	2	2	2	3	16
VI	0	0	0	0	0	0	1	4	6	7	18
VII	0	0	1	1	0	1	0	1	2	3	9
VIII	0	0	2	2	0	1	0	1	3	3	12
IX	0	1	1	1	0	1	2	2	2	3	13
X	0	0	2	3	1	1	0	1	2	3	13
XI	0	0	2	3	1	1	0	1	3	3	14

We now set $a_6 = 0$ and have

Table V

	21^3	32	41	1	62	2	3	54	4	SUM
I	0	0	2	2	1	2	1	4	6	18
II	0	1	2	2	1	2	2	2	3	15
III	0	0	2	4	2	2	1	4	6	22
IV	1	1	1	2	0	1	2	2	2	12
V	1	1	1	2	0	2	2	2	2	13
VI	0	0	0	0	0	0	1	4	6	11
VII	0	0	1	1	0	1	0	1	2	6
VIII	0	0	2	2	0	1	0	1	3	9
IX	0	1	1	1	0	1	2	2	2	10
X	0	0	2	3	1	1	0	1	2	10
XI	0	0	2	3	1	1	0	1	3	11

And this is already in quadratic form, since $L_j > R_k$ for all j and k. We can also verify that if we set $a_6 = \Delta_4 = \Delta_{54} = 0$ the function will still be in quadratic form, since SUM will be equal to (with $a_6 = \Delta_4 = \Delta_{54} = 0$).

Table VI

Vertex	21^3	32	41	1	62	2	3	SUM
I	0	0	2	2	1	2	1	8
II	0	1	2	2	1	2	2	10
III	0	0	3	4	2	2	1	12
IV	1	1	1	2	0	1	2	8
V	1	1	1	2	0	2	2	9
VI	0	0	0	0	0	0	1	1
VII	0	0	1	1	0	1	0	3
VIII	0	0	2	2	0	1	0	5
IX	0	1	1	1	0	1	2	6
X	0	0	2	3	1	1	0	7
XI	0	0	2	3	1	1	0	7

We have then

$$a_6 = \Delta_4 = \Delta_{54} = 0$$

and setting $\Delta_i = 1$

$$a_3 + a_{54} + \Delta_3 = 0+1 = 1$$

$$a_2 = a_3 + \Delta_2 = 1+1 = 2$$

$$a_{62} = a_2 + \Delta_{62} = 2+1 = 3 = a_{43} = a_{53}$$

$$a_1 = a_{62} + \Delta_1 = 3+1 = 4$$

$$a_{41} = a_{51} = a_{61} = a_{42} = a_5 = a_{52} = a_1 + \Delta_{41} = 4+1 = 5$$

$$a_{32} = a_{41} + \Delta_{32} = 5+1 = 6$$

$$\begin{aligned} a_{21} = a_{31} = a_{32} + \Delta_{21} &= a_{32} + \Delta_{21}^2 + 2\Delta_{54} + 2\Delta_4 + 3a_6 + 2\Delta_3 \\ &= a_{32} + \Delta_{21}^3 + \Delta_2 + 2\Delta_{54} + 2\Delta_4 + 3a_6 + 2\Delta_3 \\ &= 6 + 1 + 1 + 0 + 0 + 0 + 2 = 10 \end{aligned}$$

The matrix is then

$$A = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 10 & 2 & 0 & 0 & 0 & 0 \\ 10 & 6 & 1 & 0 & 0 & 0 \\ 5 & 5 & 3 & 0 & 0 & 0 \\ 5 & 5 & 3 & 0 & 0 & 0 \\ 5 & 3 & 1 & 0 & 0 & 0 \end{pmatrix}$$

and $T = \min_j (X_j^T A X_j)$ $(j=1,p')$; $T = 15.$

CONCLUSIONS

When trying to find a function F which describes the behaviour of a system depending upon several variables, it is natural to express it as

$$F = F_o + \left(\frac{\partial F}{\partial x}\right)_o x + \left(\frac{\partial^2 F}{\partial x_i \partial x_j}\right)_o x_i x_j \quad .$$

The linear part falls under the category of linear threshold functions and has been investigated in detail.

The quadratic part of the Taylor's series expansion is then partially covered by our work.

We have seen some of the properties of unate boolean functions and developed a method for realizing them. We feel that the systematic realization of non-unate boolean function should be investigated in more detail, and that would complete our work.

REFERENCES

[1] Adam, A. (1968) - Truth functions and the problem of their realization by two-terminal graphs. Akademiai Kiado, Budapest.

[2] Albert, A. (1949) - Solid analytic geometry. McGraw-Hill.

[3] Birkhoff, G., MacLane, S. (1965) - A brief survey of modern algebra. McMillan Co., New York.

[4] Arnold, B.H. (1962) - Logic and Boolean algebra. Prentice Hall.

[5] Lewis and Coates (1967) - Threshold Logic. John Wiley & Sons, New York.

[6] Dertouzos, M.L. (1964) - An approach to single threshold element synthesis. I.E.E.E. transactions on electronic computers, EC13, 519-528.

[7] Flegg, H.C. (1964) - Boolean algebra and its application. Blackie & Son Limited, London, England.

[8] Hadley, G. (1962) - Linear programming. Addison Wesley Publishing Co.

[9] Highleyman, H. (1961) - A note on linear separation. I.R.E. transactions on electronic computers, EC10, 777-778.

[10] Hruz, B. (1969) - Unateness test of a boolean function and two general synthesis method using threshold logic elements. I.E.E.E. transactions on computers, Vol. C-18, 122-131.

[11] Hu, S.T. (1968) - Threshold Logic. University of California Press.

[12] McNaughton, R. (1961) - Unate truth functions. I.E.E.E. transactions on electronic computers, EC10, 10-6.

[13] Nilsson, N.J. (1965) - Learning machines: foundations of trainable pattern-classifying systems. McGraw-Hill Company.

[14] Sheng, C.L. (1969) - Threshold Logic. The Ryerson Press, Toronto.

[15] Winder, R.O. (1962) - Threshold Logic. Doctoral Dissertation, Mathematics Department, Princeton University.